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Torus period coefficients on PGL(2) and Dirichlet series ☆

Andre Reznikov

Department of Mathematics, Bar-Ilan University, Ramat Gan 52900, Israel

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ABSTRACT

We fix a quadratic number field E, and the corresponding torus in PGL(2). We consider twisted (by a Hecke character of the field E) torus periods of automorphic functions. We prove meromorphic continuation for a Dirichlet series generated by these twisted periods.

1. Introduction

1.1. In this paper, we consider Hecke characters of a quadratic field. Although arguments presented here are valid for a general quadratic number field E, in order to simplify the presentation, we will first deal with the simplest case of Gauss numbers, and describe the general case in Section 3.2. Hence, let $E = \mathbb{Q}(i)$. For an integer $n \in \mathbb{Z}$, consider the Hecke character of E given by $\chi_n(a) = (a/|a|)^{4n}$. The corresponding Hecke L-function is given by the series $L(s,\chi) = \sum_{a \in I^*(\mathcal{O}_E)} \chi(a) N(a)^{-s}$, for $Re(s) > 1$ (where the summation is over all non-zero integer ideals of E, i.e., over $\mathbb{Z}[i]/\{\pm 1, \pm i\}$ for $E = \mathbb{Q}(i)$).

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E-mail address: reznikov@math.biu.ac.il.

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We consider a double Dirichlet series given by

$$D_E(s,w) = L(s,\chi_0) + \sum_{n\in\mathbb{Z}\setminus 0} L(s,\chi_n)|n|^{-w}, \tag{1.1}$$

for $(s,w)\in\mathbb{C}^2$, $Re(s)\gg 1$, $Re(w)\gg 1$.

Theorem 1.2. *The series $D_E(s,w)$ defines the function which extends to a meromorphic function on $\mathbb{C}^2$.*

It turns out that it is more convenient to consider the function

$$\tilde{D}_E(s,w) = \frac{2^{\frac{w}{2}}}{\Gamma(\frac{1-w}{2})}\cdot D_E(s,w), \tag{1.2}$$

and we prove the meromorphic continuation for this function.

1.3. Torus periods

One quickly recognizes that the above theorem is related to periods of automorphic functions. In fact, our proof of the meromorphic continuation is based on two well-known facts. First, we invoke classical results of E. Hecke [H] (see also H. Maass [M] and C. Siegel [Si]) about torus periods of Eisenstein series. Namely, we consider the automorphic representation Eis_s generated by the (normalized) Eisenstein series $E_s(z)$ for PGL_2 over $\mathbb{Q}$. Let $T_E\subset \mathrm{PGL}_2$ be the torus corresponding to E, and $\bar{e}\in T_E(\mathbb{Q})\setminus T_E(\mathbb{A}_\mathbb{Q})$ be the class of the identity element. The (Fourier) expansion of the Eisenstein series $E_s(z)$ along the orbit $T_E(\mathbb{Q})\setminus T_E(\mathbb{A}_\mathbb{Q})\subset \mathrm{PGL}_2(\mathbb{Q})\setminus \mathrm{PGL}_2(\mathbb{A}_\mathbb{Q})$ is given in terms of Hecke characters of E, and naturally leads to Hecke L-functions $L(s,\chi)$ for Hecke characters χ of the field E. This allows us to realize the series $D_E(s,w)$ as the spectral expansion of $E_s(v_w,\bar{e})$, $w\in\mathbb{C}$, for the function $E_s(v_w,x)\in Eis_s$ corresponding to some special vector v_w. The family of vectors v_w is constructed explicitly in the induced model of the representation isomorphic to Eis_s. We note that the vector v_w is not smooth, and belongs to an appropriate Sobolev space completion of Eis_s. In particular, we invoke the meromorphic continuation of smooth Eisenstein series as opposed to K-finite Eisenstein series (for a general treatment, see [BK,L]; for a congruence subgroup of $\mathrm{PGL}_2(\mathbb{Z})$, an elementary treatment based on Fourier expansion of $E_s(z)$ is also available).

To prove the meromorphic continuation of $E_s(v_w,\bar{e})$, we use Hecke operators and the classical technique going back to at least M. Riesz [R] (and might be attributed to Euler) of the analytic continuation strip by strip (which the author learned from the seminal paper [B]). The main observation that allows us to apply this technique is the fact that modulo higher Sobolev spaces, the vector v_w is an approximate eigenvector of (appropriately understood) Hecke operators (see Lemma 2.5). This is verified in a standard model of an abstract representation of the principal series, and does not use the theory of automorphic functions.

Finally, we would like to point out that for the method we use, the meromorphic continuation of $E_s(v_w,\bar{e})$ is what comes naturally, and the series $D_E(s,w)$ is used in order to express this fact in the classical language of automorphic functions on $\mathcal{H}$.

1.4. Cusp forms

One can apply the same argument to a Hecke–Maass cusp form ϕ instead of the Eisenstein series $E(s)$. The resulting series is a usual Dirichlet series in *one variable* built from the coefficients a_n which are (twisted) torus periods of the cusp form ϕ. We now recall the definition of these coefficients.

Let ϕ be a Hecke–Maass form for the group $\Gamma = \mathrm{PGL}_2(\mathbb{Z})$ (one can easily extend our arguments to a congruence subgroup). In particular, ϕ is an eigenfunction of the Laplace–Beltrami operator Δ on the Riemann surface $\mathrm{PSL}_2(\mathbb{Z})\setminus\mathcal{H}$ with the eigenvalue which we denote by $\Lambda(\phi) = (1-\tau^2)/4$

for $\tau = \tau(\phi) \in i\mathbb{R} \cup (0,1)$ (of course, for $\mathrm{PGL}_2(\mathbb{Z})$, the parameter τ is pure imaginary, and this is expected to hold for congruence subgroups). We normalize ϕ by its L^2-norm. We denote by (V_τ, π_τ) the isomorphism class of the (smooth) automorphic representation of $G = \mathrm{PGL}_2(\mathbb{R})$ generated by ϕ. The structure of such a representation is well-known, and in particular V_τ has an orthonormal basis of K-types $\{e_n\}_{n\in 2\mathbb{Z}}$ which we fix (here $K = \mathrm{PSO}(2,\mathbb{R}) \simeq S^1$ is a maximal connected compact subgroup of G). We consider the Taylor-like expansion of ϕ at $z = i$ (generally we consider a CM-point $\mathfrak{z} \in \mathcal{H}$; in fact such an expansion exists at any point of $\mathcal{H}$). Denote by $F_\tau(n,g) = \langle \pi_\tau(g)e_0, e_n\rangle_{\pi_\tau}$, $n \in 2\mathbb{Z}$, matrix coefficients in the representation π_τ. Functions $F_\tau(n,g)$ are right K-invariant and hence could be viewed as functions of $z \in \mathcal{H} \simeq \mathrm{PGL}_2^+(\mathbb{R})/K$. Functions $F_\tau(n,z)$ are eigenfunctions of Δ on $\mathcal{H}$ with the same eigenvalue as ϕ, and spherically equivariant $F_\tau\left(n, \left(\begin{smallmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{smallmatrix}\right) z\right) = e^{in\theta} F_\tau(n,z)$. It is also well-known that functions $F_\tau(n,z)$ have simple integral representation and could be expressed through the Legendre function (see [Vi]). We have the following well-known expansion (first considered by H. Petersson [P] for holomorphic forms and later by A. Good [Go] in general; also see [Sa]):

$$\phi(z) = \sum_{n\in 2\mathbb{Z}} a_n F_\tau(n,z), \tag{1.3}$$

where $a_n = a_n(\phi) \in \mathbb{C}$. Of course, coefficients a_n depend on the normalization of functions $F_\tau(n,z)$, which we fix in Section 2.3.3 (by choosing a basis $\{e_n\}$ of π_τ). This normalization will essentially coincide with one of the classical normalizations of the special function $F_\tau(n,z)$. In particular, this will not depend on ϕ but only on the parameter τ. We note that the analogous expansion is valid for the Eisenstein series $E(s,z)$ as well, and gives $a_n(E(s)) = L(s, \chi_{-n})$.

We consider the Dirichlet series

$$D_E(\phi, w) = a_0 + \sum_{n\in 2\mathbb{Z}\setminus 0} a_n |n|^{-w} \tag{1.4}$$

defined for $|w| \gg 1$. As with the Eisenstein series we consider the function

$$\tilde{D}_E(\phi, w) = \frac{2^{\frac{w}{2}}}{\Gamma(\frac{1-w}{2})} \cdot D_E(\phi, w). \tag{1.5}$$

Theorem 1.5. *The Dirichlet series $\tilde{D}_E(\phi, w)$ extends to a holomorphic function on $\mathbb{C}$.*

1.6. Hyperbolic periods

A similar treatment is available for real quadratic fields. These correspond to (compact) closed geodesics on the Riemann surface $Y = \mathrm{PGL}_2(\mathbb{Z}) \setminus \mathrm{PGL}_2(\mathbb{R})$. In fact, from the adelic point of view, there is no difference in the treatment of CM-points and of closed geodesics.

Let $l \subset Y$ be a closed geodesic. Such a geodesic corresponds to a closed orbit of the diagonal subgroup $A = \{\mathrm{diag}(a,b)\} \subset G$ acting on the right on $X = \Gamma \setminus G$. We denote this orbit by the same letter $l \subset X$. Consider the corresponding pointwise stabilizer $A_l = \mathrm{Stab}_A(l)$. We will assume that it is cyclic and we will choose a generator $a_l = \mathrm{diag}(u_l, \pm u_l^{-1}) \in A_l$. This choice gives the corresponding hyperbolic element $\gamma_l \in \Gamma$ which is conjugate to $a_l = g_l \gamma_l g_l^{-1}$. Eigenvalues of a_l (and of γ_l) generate the group of units in a quadratic field E. In fact, there is a finite number of closed geodesics corresponding to the same field and this is reflected in the class number of the field.

For a closed geodesic l as above, we obtain an expansion of automorphic functions similar to the expansion at a CM-point we discussed above. Such an expansion is valid for cusp forms and for Eisenstein series (e.g., see [Gol]). To describe these in classical terms, one introduces special functions similar to $F_\tau(n,z)$ above. This time we use characters $\chi : A_l \setminus A \to \mathbb{C}^\times$ of the compact group $A_l \setminus A \simeq S^1$. For any such a character, we consider the function $G_\tau(\chi, z)$ on $\mathcal{H}$ which is an eigenfunction of Δ, is right K-invariant, and satisfies $G_\tau(\chi, g_l^{-1} a g_l \circ z) = \chi(a) G_\tau(\chi, z)$. In fact, we can view the function $G_\tau(\chi, z)$ as defined on the hyperbolic cylinder $\mathcal{H}_l = \Gamma_l \setminus \mathcal{H}$ for which l is the "neck" (i.e., the shortest geodesic). We have to normalize functions $G_\tau(\chi, z)$ which we do in Section 2.3.3 by presenting an explicit integral representation for these functions (the function $G_\tau(\chi, z)$ classically is given in terms of the Gauss hypergeometric function as could be seen from its explicit integral representation as a matrix coefficient). We obtain the expansion analogous to (1.3)

$$\phi(z) = \sum_{j\in\mathbb{Z}} b_j G_\tau(j, z), \tag{1.6}$$

where $b_j = b_{\chi_j}(\phi) \in \mathbb{C}$ are the coefficients of this hyperbolic expansion, and functions $G_\tau(j, z) = G_\tau(\chi_j, z)$ are indexed by characters χ_j in the group $\widehat{A_l \setminus A} \simeq \mathbb{Z}$. With this for $|w| \gg 1$, we define the Dirichlet series

$$D_E(\phi, w) = b_0 + \sum_{j\in\mathbb{Z}\setminus 0} b_j |j|^{-w}. \tag{1.7}$$

As before, we consider the function

$$\tilde{D}_E(\phi, w) = \frac{2^{\frac{w}{2}}}{\Gamma(\frac{1-w}{2})} \cdot D_E(\phi, w). \tag{1.8}$$

Theorem 1.7. *The Dirichlet series $\tilde{D}_E(\phi, w)$ extends to a holomorphic function on $\mathbb{C}$.*

We hope that by denoting the series by the same symbol as in the CM-case, we will not cause too much of a confusion (and in fact from the adelic point of view the treatment of these two cases is identical).

1.8. Remarks

(1) In definitions (1.1), (1.4) and (1.7) of the corresponding Dirichlet series one clearly can omit the zero's term in the sum. One also can take the sum over positive n only. Our method is applicable to such series as well.

(2) Under the normalization, we chose in (1.3), coefficients a_n satisfy a mean-value bound $c_\phi \leqslant T^{-1} \sum_{|n|\leqslant T} |a_n|^2 \leqslant C_\phi$ for appropriate constants $C_\phi, c_\phi > 0$ as $T \to \infty$. Hence coefficients a_n are not exponentially small on the average, and this is not a reason behind the meromorphic continuation of the series $D_E(\phi, w)$. The same is true for b_n's in (1.6).

(3) Coefficients a_n and b_n are related to L-functions in a more subtle way than for the Eisenstein series. Namely, the theorem of J.-L. Waldspurger (see [W,JC,KW]) relates the value of $|a_n|^2$ and of $|b_n|^2$ to the ratio of L-functions $L(1/2, BC_E(\phi) \otimes \chi_n)/L(1, Ad(\phi))$, where $BC_E(\phi)$ is the base change of the cusp form ϕ. In spite of this relation, our method naturally treats coefficients a_n and not quantities $|a_n|^2$.

(4) The proof that we give shows that the polar divisor of $D_E(s, w)$ is contained in the union of the line $s = 1$ with the union of two families of lines

$$w = 2 - 2s - j, \quad \text{or} \quad w = 1 - j, \quad j = 0, 1, 2, \ldots. \tag{1.9}$$

A somewhat more symmetric form of $\tilde{D}_E(s, w)$ and of its polar set is discussed in Remark 3.1. For a Hecke–Maass cusp form ϕ, we show that the series $\tilde{D}_E(\phi, w)$ is holomorphic. One can also obtain polynomial bounds in s and w for the resulting function $\tilde{D}_E(s, w)$.

After announcing the results of the paper, the author was kindly informed by the referee that one can deduce the meromorphic continuation, and the exact locations of poles for the series $D_E(s,w)$ from properties of the Lerch zeta function (see [LG]), and from the functional equation for the Hecke L-functions $L(s,\chi)$. While this approach is more elementary, it could not cover the case of cusp forms. On the other hand, for the Eisenstein series, this approach leads to the exact location of poles, while our method only gives the potential polar divisor of $D_E(s,w)$.

It is also apparently possible to deduce meromorphic continuation of $D_E(\phi,w)$ for cusp forms which are not necessarily Hecke forms, using methods of [BLZ], as was demonstrated by R. Bruggeman (personal communication). It is however less clear how to extend this approach to cover real quadratic fields or the Eisenstein series.

(5) The set of all Hecke characters $\chi_n(a)=(a/|a|)^{4n}$, $n\in\mathbb{Z}$, of $E=\mathbb{Q}(i)$ could be described as the set $\mathcal{X}_{un}(E)$ of all (maximally) unramified Hecke characters of E. One can consider a slightly more general series by prescribing the ramification of Hecke characters. Let S be a finite set of primes of a quadratic field E, including all primes ramified in E, and let $\mathcal{X}_S$ be the set of Hecke characters unramified outside S. The natural extension of our method then gives the meromorphic continuation for the series $\sum_{\chi\in\mathcal{X}_S} L_S(s,\chi)R_S(\chi^S)|\chi_\infty|^{-w}$, where $L_S(s,\chi)$ is the partial L-function, $R_S(\chi^S)$ is a rational in q_i^s function for q_i that are norms of primes in S, and χ^S is the ramified part of χ.

(6) An important issue in the theory of (double) Dirichlet series is the presence of functional equation(s). The theory of Eisenstein series provides the functional equation in s relating $D_E(s,w)$ and $D_E(1-s,w)$. It is not clear if there is a functional equation involving w.

(7) Finally, we note that from the point of view of the method we present, there is nothing special about series $D_E(s,w)$ and $D_E(\phi,w)$. Namely, one can change the weight $|n|^{-w}$ to many other similar weight functions, and still obtain the meromorphic continuation by the same method. As a result, it is possible that one might have to modify these series in order to study their possible arithmetical properties (e.g., special values).

2. Torus periods

We refer to [Bu] for standard facts about automorphic functions and automorphic representations (of real and adele groups).

2.1. Torus periods of Eisenstein series

We recall the classical result of E. Hecke. We present it in (a more transparent to us) adelic language. Let $G=\mathrm{PGL}_2$. By specifying an isomorphism $E^\times\subset Aut_{\mathbb{Q}}(E)\simeq Aut_{\mathbb{Q}}(\mathbb{Q}^2)$, we obtain the corresponding tori $T_E\subset\mathrm{PGL}_2$ defined over $\mathbb{Q}$. For $s\in\mathbb{C}$, $s\neq 1$, let $\mathcal{E}_s\simeq\hat{\bigotimes}_{p\leqslant\infty}E_{s,p}$ be the automorphic representation of $G_{\mathbb{A}_{\mathbb{Q}}}$ corresponding to the classical *normalized* Eisenstein series (given by $E_s(z)=\sum_{c,d}y^s/|cz+d|^{2s}$ for $Re(s)>1$, where the summation is over $(c,d)\in\mathbb{Z}^2\setminus(0,0)$). In this normalization, the unitary Eisenstein series corresponds to $Re(s)=1/2$. We consider vectors in $\mathcal{E}$ which are pure tensors of the form $v_\infty\otimes v_f\in\mathcal{E}_s$ where $v_f=\bigotimes_{p<\infty}v_p$ is the standard K_f-fixed vector for the maximal compact subgroup K_f of G over finite adeles, and v_∞ is an arbitrary vector in the infinite component $E_{s,\infty}$ of $\mathcal{E}_s$. Recall that the theory of Eisenstein series provides the automorphic realization $E_s(v,g)$ (i.e., a function on $X_{\mathbb{A}}$) for a vector $v\in\mathcal{D}_s$ in the principal series representation $\mathcal{D}_s$ of $G_{\mathbb{A}_{\mathbb{Q}}}$, where $\mathcal{D}_s$ is the space of homogeneous functions with respect to the $\mathbb{A}_{\mathbb{Q}}^\times$ action on the space $Z_{\mathbb{A}_{\mathbb{Q}}}=N_{\mathbb{A}_{\mathbb{Q}}}\setminus G_{\mathbb{A}_{\mathbb{Q}}}=\prod'_{p\leqslant\infty}N_{\mathbb{Q}_p}\setminus G_{\mathbb{Q}_p}$. The space $\mathcal{D}_s$ has the natural structure of the (restricted) tensor product $\mathcal{D}_s\simeq\hat{\bigotimes}_{p\leqslant\infty}D_{s,p}$ coming from the above product structure of $Z_{\mathbb{A}_{\mathbb{Q}}}$ (unlike the space $\mathcal{E}_s$ where the tensor product is not described in terms of the underlying space $X_{\mathbb{A}}=G_{\mathbb{Q}}\setminus G_{\mathbb{A}}$). Here local components are the spaces $D_{s,p}$ of homogeneous functions on $N_p\setminus G_p$. Hence when talking about models of the local representations $E_{s,p}$, we can use the spaces $D_{s,p}$ (in fact, we only use $p=\infty$ since we will not discuss ramified Hecke characters).

According to Hecke (via the standard by now, unfolding) (see [G,Gol]) we have the following relation. Let $v=v_\infty\otimes v_f\in\mathcal{E}_s$ be a vector with almost everywhere unramified standard finite components. We have then

$$\int_{T_E(\mathbb{Q})\backslash T_E(\mathbb{A}_\mathbb{Q})} E_s(v,t)\chi(t)\,dt = I_\infty(s,\chi_\infty,v_\infty)\prod_{p\in S} I_\nu(s,\chi_p,v_p)\cdot L_S(s,\chi), \tag{2.1}$$

for any character $\chi=\chi_\infty\otimes\chi_f : Z_G(\mathbb{A}_\mathbb{Q})T_E(\mathbb{Q})\setminus T_E(\mathbb{A}_\mathbb{Q})\to\mathbb{C}^\times$ (i.e., for a Hecke character of E trivial on $\mathbb{A}_\mathbb{Q}^\times\subset\mathbb{A}_E^\times$). Here the functional $I_\infty(s,\chi_\infty,\cdot):E_{s,\infty}\to\mathbb{C}$ is given by

$$I_\infty(s,\chi_\infty,v_\infty)=\int_{T_E(\mathbb{R})} v_\infty(t)\chi_\infty(t)\,dt, \tag{2.2}$$

for a vector $v\in D_{s,\infty}$, and similarly for ramified primes $p\in S$. $L_S(s,\chi)$ denotes the partial L-function with Euler factors removed at ramified primes.

Note that for $E=\mathbb{Q}(i)$, the group $T_E(\mathbb{R})$ could be naturally identified with the subgroup $K_\infty=\mathrm{PSO}(2,\mathbb{R})\subset\mathrm{PGL}_2(\mathbb{R})$ (i.e., $K_\infty=\mathrm{PSO}(2,\mathbb{R})$ is the standard maximal connected compact subgroup of $G_\infty=\mathrm{PGL}_2(\mathbb{R})$). Hence for a Hecke character χ_n, the resulting functional $I_\infty(s,\chi_{n,\infty},\cdot)$ could be identified with the projection to the particular norm one n-th K_∞-type in the representation $E_{s,p}$. Moreover, in the realization of $E_{s,\infty}$ as $D_{s,\infty}$, this functional is given by the integration against the character *itself* on the image of the compact subgroup $K_\infty\subset Z_\infty=N_\infty\setminus G_\infty$ coming from the archimedean part $(T_E)_\infty\subset G_\infty$ of the torus T_E. For other CM-fields, we obtain a compact subgroup conjugated to $\mathrm{PSO}(2,\mathbb{R})$, and hence have to consider types with respect to the corresponding subgroup.

By abuse of notations, we denote by $e\in G_\mathbb{Q}\setminus G_{\mathbb{A}_\mathbb{Q}}$ the image of the identity. From the Plancherel formula for $T_E(\mathbb{Q})\setminus T_E(\mathbb{A}_\mathbb{Q})$ (i.e., the Fourier expansion w.r.t. characters of $T_E(\mathbb{A}_\mathbb{Q})$ trivial on $T_E(\mathbb{Q})$), and the Hecke formula (2.1), we see that the following expansion holds for a vector $v=v_\infty\otimes v_f\in\mathcal{E}_s$:

$$E_s(v,e)=\sum_{\chi_n\in\mathcal{X}_{un}} L(s,\chi_n)\cdot I_\infty(s,\chi_{n,\infty},v_\infty). \tag{2.3}$$

2.2. *Periods of cusp forms*

Periods of cusp forms could be defined in the same way as for the Eisenstein series. However there is an important difference concerning their normalization.

Let $\pi\simeq\hat{\bigotimes}_{p\leqslant\infty}\pi_p$ be an automorphic cuspidal representation of $G(\mathbb{A})$ in the space of smooth vectors $V_\pi\simeq\hat{\bigotimes}_{p\leqslant\infty}V_{\pi_p}$, together with the isometric realization $\nu_\pi:V_\pi\to C^\infty(X_\mathbb{A})$. Let $T_E(\mathbb{Q})\setminus T_E(\mathbb{A}_\mathbb{Q})\subset\mathrm{PGL}_2(\mathbb{Q})\setminus\mathrm{PGL}_2(\mathbb{A}_\mathbb{Q})$ be the orbit of $T_E(\mathbb{A})$. For a Hecke character χ of E, we consider the corresponding χ-equivariant functional $d_\chi^{aut}\in\mathrm{Hom}_{T_E(\mathbb{A})}(\pi,\chi)$ given by the integral (as for the Eisenstein series)

$$d_\chi^{aut}(v)=\int_{T_E(\mathbb{Q})\backslash T_E(\mathbb{A}_\mathbb{Q})}\phi_v(t)\bar{\chi}(t)\,dt, \tag{2.4}$$

where $\phi_v=\nu_\pi(v)\in C^\infty(X)$ is the automorphic function corresponding to the smooth vector $v\in V_\pi$ under the isometry ν_π (i.e., the automorphic realization of the vector v). It is well-know that the space of local equivariant functionals is at most one-dimensional $\dim\mathrm{Hom}_{T_p}(\pi_p,\chi_p)\leqslant 1$, and hence we have a decomposition $d_\chi^{aut}=\bigotimes_p d_{\chi_p}$ for some local functionals $d_{\chi_p}\in\mathrm{Hom}_{T_p}(\pi_p,\chi_p)$. However, unlike in the case of Eisenstein series, the lack of unfolding for the integral (2.4) does not allow us to choose easily a specific element in the space $\mathrm{Hom}_{T_p}(\pi_p,\chi_p)$. In fact, one can normalize d_{χ_p} up to a constant with absolute value one (this is connected to the Waldspurger theorem alluded before, and discussed in great generality in [II]), but for our purposes it is not enough since we are interested in the period itself and not in its absolute value. Hence, we will choose the trivial normalization of local functionals in the following way.

We assume that the representation π is unramified everywhere, i.e., that the Hecke–Maass form ϕ is invariant under the full group $\mathrm{PSL}_2(\mathbb{Z})$ (in fact, we can easily deal with any congruence subgroup). The notion of restricted tensor product assumes that we have chosen a K_p-invariant vector $e_p \in V_{\pi_p}$ of norm one for every finite p (in general for almost all p). We have $\dim \mathrm{Hom}_{T_p}(\pi_p, \chi_p) = 1$ since we assumed that π_p is unramified for all p. It is known (see [GP]) that a non-zero invariant functional does not vanish on the vector e_p. We denote by $d^{mod}_{\chi_p} \in \mathrm{Hom}_{T_p}(\pi_p, \chi_p)$ the functional satisfying $d^{mod}_{\chi_p}(e_p) = 1$. We consider the corresponding functional $d^{mod}_{\chi_f} = \bigotimes_{p<\infty} d^{mod}_{\chi_p}$ for finite adeles which clearly satisfies $d^{mod}_{\chi_f}(e_f) = 1$ for $e_f = \bigotimes_{p<\infty} e_p$. Hence for any choice of a non-zero functional $d^{mod}_{\chi_\infty} \in \mathrm{Hom}_{T_\infty}(\pi_\infty, \chi_\infty)$, we obtain the coefficient of proportionality $a_\chi = a_\chi(\nu_\pi, d_{\chi_\infty}) \in \mathbb{C}$ such that

$$d^{aut}_\chi = a_\chi \cdot d^{mod}_{\chi_\infty} \otimes d^{mod}_f. \tag{2.5}$$

In fact, since we will only consider vectors of the form $v = v_\infty \otimes e_f$, we can write $d^{aut}_\chi(v_\infty) = a_\chi \cdot d^{mod}_{\chi_\infty}(v_\infty)$. We now specify the functional $d^{mod}_{\chi_\infty}$. As we mentioned, for $E = \mathbb{Q}(i)$, we can naturally identify $T_\infty = T_E(\mathbb{R})$ with the subgroup K_∞. Characters of T_∞ related to equivariant functionals on irreducible representations of G_∞ are parameterized by even integers $n \in 2\mathbb{Z}$, and naturally correspond to projectors onto (one-dimensional) K_∞-types. Let $\{e_n\}_{n\in 2\mathbb{Z}}$ be an orthonormal basis of V_{π_∞} consisting of K_∞-types. We denote by $d^{mod}_n \in \mathrm{Hom}_{T_\infty}(\pi_\infty, \chi_n)$ the functional given by $d^{mod}_n(v_\infty) = \langle v, e_n\rangle_{\pi_\infty}$, where the character χ_n is given by $\chi_n\left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right) = e^{in\theta}$.

Hence we have the decomposition analogous to (2.3)

$$\phi_{v_\infty}(e) := \nu_\pi(v_\infty)(e) = \sum_{\chi_n \in \mathcal{X}_{un}} a_n \cdot d^{mod}_n(v_\infty). \tag{2.6}$$

It is easy to see that this expansion coincides with the expansion (1.3) given in classical terms on $\mathcal{H}$.

2.3. *Test vectors*

In order to realize the series $D_E(s, w)$ and $D_E(\phi, w)$ as the right hand side of formulas (2.3) and (2.6), we need to construct a vector v_w in the principal series representation satisfying certain properties. We construct such a vector and make computations in a well-known model of induced representations of $\mathrm{PGL}_2(\mathbb{R})$.

2.3.1. *The plane realization*

The basic affine space Z_∞ is isomorphic to the punctured plane $\mathbb{R}^2 \setminus 0$. This leads to the standard realization of the principal series representation in even homogeneous functions on the plane. For a complex parameter $\tau \in \mathbb{C}$ and $\varepsilon \in \{0, 1\}$, the (smooth part of the) representation $\pi_{\tau,\varepsilon}$ of principal series has the realization in the space of homogeneous functions on $\mathbb{R}^2 \setminus 0$ of the homogeneous degree $\tau - 1$. The twisted action $\pi_{\tau,\varepsilon}(g)f(t) = f(g^{-1}t)|\det g|^{\frac{\tau-1}{2}} \det(g)^\varepsilon$, $t \in \mathbb{R}^2 \setminus 0$, defines a representation of $\mathrm{GL}_2(\mathbb{R})$ with the trivial center character, and hence defines a representation of $\mathrm{PGL}_2(\mathbb{R})$. The trivial representation is the subrepresentation for $\tau = 1$, $\varepsilon = 0$ (and the quotient for $\tau = -1$). The standard Casimir operator acts on $\pi_{\tau,\varepsilon}$ by multiplication by the scalar $\frac{1-\tau^2}{4}$. The dual representation to $\pi_{\tau,\varepsilon}$ could be naturally identified with $\pi_{-\tau,\varepsilon}$. Representations $\pi_{\tau,\varepsilon}$ are unitarizable for $\tau \in i\mathbb{R} \cup (-1, 1)$.

Taking the restriction of functions on $\mathbb{R}^2 \setminus 0$ to the circle $S^1 \subset \mathbb{R}^2 \setminus 0$, we obtain the circle (or compact) model for the space of $\pi_{\tau,\varepsilon}$. This means that we realize the space of the representation as the space of smooth even functions $C^\infty_{ev}(S^1)$ on the circle S^1 (or on $K_\infty \simeq S^1$). Hence in such a model, a K_∞-equivariant functional is given by the integration against the exponent $e^{in\theta}$, i.e., the scalar product with a norm one n-th K_∞-type.

Taking the restriction of functions on the plane to a line $L \subset \mathbb{R}^2 \setminus 0$, we obtain a line (or unipotent) model for the space of $\pi_{\tau,\varepsilon}$.

An easy computation shows that in the above described normalization of the principal series and the identification $Z_\infty \simeq \mathbb{R}^2 \setminus 0$, the infinity component $E_{s,\infty}$ is isomorphic to the representation of the principal series with the parameter $\tau = 1 - 2s$.

In what follows we will treat "even" representations only (i.e., $\varepsilon = 0$). The treatment of "odd" representations is identical. Hence in what follows, we denote representations of $\mathrm{PGL}_2(\mathbb{R})$ by π_τ suppressing ε. We note that, for $\mathrm{PSL}_2(\mathbb{Z})$, representations appearing as Eisenstein series are automatically even.

2.3.2. Test vectors

In order to realize the series $D_E(s,w)$ (and the corresponding series $D_E(\phi,w)$), we construct the test vector v_w in the representation of the principal series π_{1-2s} (respectively in π_τ) of $G_\infty = \mathrm{PGL}_2(\mathbb{R})$ with the K_∞-types components satisfying $\hat{v}_w(n) := \langle v_w, e_n\rangle = |n|^{-w}$ for an even integer $n \neq 0$, and $\hat{v}_w(0) = 1$. Clearly, such a vector exists in an appropriate completion of the corresponding smooth representation. For the unitary principal series π_τ, the vector v_w belongs to the L^2-Sobolev space $S_\sigma(\pi_\tau)$ of index $\sigma = Re(w) - 1/2$ (see [BR]). Moreover, it is easy to see that such a vector has "local" singularities in the natural spherical model of the representation. This fact is central for our approach. We now describe the construction and the structure of singularities of the test vector.

The (smooth part of the) representation π_{1-2s} of principal series has the above mentioned realization in the space $C^\infty_{ev}(S^1)$ of smooth even functions on the circle S^1. We denote by θ the parameter on S^1 (e.g., $\theta \in [0, 2\pi)$), and by $e_n(\theta) = e^{in\theta}$ the standard orthonormal basis. For a smooth even function $f \in C^\infty_{ev}(S^1)$, we denote by $\hat{f}(n)$ its Fourier coefficients. This defines the isometry $\hat{}: L^2_{ev}(S^1) \to L^2(\mathbb{Z})$. Hence for any $w \in \mathbb{C}$ with $Re(w) > 1/2$, we have a function $v_w \in L^2_{ev}(S^1)$ such that $\hat{v}_w(n) = |n|^{-w}$ for $n \neq 0$, and $\hat{v}_w(0) = 1$. For $w \leqslant 1/2$, we should view v_w as a distribution on $C^\infty_{ev}(S^1)$. It turns out that it is not convenient to work directly with the vector v_w since it does not "localize" (i.e., it is not supported in a small neighborhood of $\theta = 0$, the fixed point of a Borel subgroup). Instead we construct a vector u_w (i.e., a function in $C^\infty_{ev}(S^1)$) which has small support and asymptotically has essentially the same Fourier coefficients as v_w.

Well-known properties of the Fourier transform suggest that the vector with a local behavior $|\theta|^{w-1}$ near $\theta = 0$ should give us the desired Fourier coefficients $|n|^{-w}$, at least for $|n| \to \infty$. It is also well-known that, as an analytic family, the distribution $\frac{2^{\frac{1-w}{2}}}{\Gamma(w/2)} \cdot |t|^{w-1}$ (on $\mathbb{R}$) and its Fourier transform $\frac{2^{w/2}}{\Gamma(\frac{1-w}{2})} \cdot |\xi|^{-w}$ behave better than the distribution $|t|^{w-1}$ and its Fourier transform (see [G1]). This explains our multiplication of the series $D_E(s,w)$ by the factor $\frac{2^{w/2}}{\Gamma(\frac{1-w}{2})}$.

Let $f \in C^\infty_{ev}(S^1)$ be a smooth even function which is supported in a small neighborhood (to be specified later) of points $\theta = 0, \pi$, and $f(\theta) \equiv 1$ in some (smaller) neighborhood of $0, \pi$. We consider the vector in the circle model given by

$$u_w(\theta) = \frac{2^{\frac{1-w}{2}}}{\Gamma(\frac{w}{2})} \cdot |\theta|^{w-1} f(\theta), \tag{2.7}$$

for $|\theta|$ near 0, and then extended to an even function on S^1 (i.e., we define u_w near $\theta = 0$ and then extend it to an even function on S^1). We have then, for an even integer $n \neq 0$,

$$\hat{u}_w(n) = \int_{S^1} u_w(\theta) e^{-in\theta}\, d\theta = \frac{2^{\frac{w}{2}}}{\Gamma(\frac{1-w}{2})} \cdot |n|^{-w}\big[1 + r(f,w,n)\big], \tag{2.8}$$

where $r(f,w,n)$ is a holomorphic function in w for every n, which is decaying at least as $|n|^{-1}$ for every fixed f and w. Moreover, the function r has an asymptotic expansion in $|n|^{-1}$ with coefficients effectively bounded in terms of w and derivatives of f. Namely, we have for any $N \geqslant 1$ and $n \neq 0$,

$$r(f,w,n)=\sum_{k=1}^{N}c_k(f,w)|n|^{-k}+r_N(f,w,n),\tag{2.9}$$

for some coefficients $c_k(f,w)$ holomorphically depending on w for a fixed f. Here the remainder satisfies the bound

$$\left|r_N(f,w,n)\right|\leqslant C_N(f,w)|n|^{-N-1},\tag{2.10}$$

for a constant $C_N(f,w)$ depending on w and f.

The relation (2.8) is valid for $Re(w)>0$, but could be extended to the whole $\mathbb{C}$ if we view the family of functions u_w as a distribution analytically depending on $w\in\mathbb{C}$ for a fixed f.

Together with the relation (2.3) and known properties of K-finite Eisenstein series (moderate growth in the type and analyticity in s), the relation (2.8) implies that

$$E_s(u_w,e)=\sum_n a_n\big(E(s)\big)\cdot\hat{u}_w(n)=\tilde{D}_E(s,w)+R(f,s,w),\tag{2.11}$$

where $R(f,s,w)=\sum_{n\neq 0}a_n(E(s))\cdot r(f,w,n)$. Here we denote by $a_n(E(s))=L(s,\chi_{-n})$ the corresponding coefficients for the Eisenstein series. This relation holds as long as $E_s(u_w,e)$ is well-defined. The theory of smooth Eisenstein series [BK,L] implies that the value at a point for an Eisenstein series for a non-K-finite vector is well-defined as long as the defining vector is smooth enough (e.g., belongs to a certain Sobolev space). In particular, $E_s(u_w,e)$ is well-defined for $Re(w)>T(s)$ with some $T(s)\in\mathbb{R}$ *depending* on s (e.g., for unitary Eisenstein series $E(s)$, $Re(s)=1/2$, we can take $T(s)=1$ although this is immaterial to us). For $Re(w)>T(s)$, the series $D_E(s,w)$ is absolutely convergent. We point out the crucial fact for us that the series $R(f,s,w)$ is absolutely convergent in a bigger domain $Re(w)>T(s)-1$ as follows from the expansion (2.9) and the bound (2.10).

Hence in order to meromorphically continue the series $\tilde{D}_E(s,w)$ (and as a result the series $D_E(s,w)$), it is enough to analytically continue $E_s(u_w,e)$. We do this strip by strip in the variable w for each *fixed* s. Analyticity in s comes from the theory of smooth Eisenstein series. We point out that for the method we use, the meromorphic continuation of $E_s(u_w,e)$ is what comes naturally. The series $D_E(s,w)$ is used in order to translate this fact into the classical language of automorphic functions on $\mathcal{H}$.

For a Hecke–Maass cusp form ϕ in a representation π_τ, the construction of the test vector u_w is identical, and we have

$$\phi_{u_w}(e)=\tilde{D}_E(\phi,w)+R(f,\nu_\pi,w),\tag{2.12}$$

where the function R is holomorphic in a bigger domain. This relation holds as long as $\phi_{u_w}(g)$ is well-defined. According to [BR], this is satisfied if the vector u_w belongs to the 1/2-Sobolev space for the representation π_τ. The last condition holds if $Re(w)>1$. Hence (2.12) is valid for $Re(w)>1$.

2.3.3. Normalization of functions $F_\tau(n,z)$ and $G_\tau(\chi,z)$

In order to construct the Dirichlet series $D_E(\phi,w)$, it seems that we have to normalize matrix coefficients $F_\tau(n,z)$, and hence the coefficients a_n in the expansion (1.3) (and similarly for the expansion in (1.6)). In fact it is not needed. Consider any orthonormal basis $\{e_n\}$ of K-types, corresponding matrix coefficients $F_\tau(n,z)=\langle\pi_\tau(g)e_0,e_n\rangle$, and the expansion (1.3) $\phi(z)=\sum_{n\in 2\mathbb{Z}}a_nF_\tau(n,z)$. We have then $D_E(\phi,w)=a_0+\sum_{n\in 2\mathbb{Z}\setminus 0}a_n\langle v_w,e_n\rangle_{\pi_\tau}$. This expression does not depend on the choice of the basis $\{e_n\}$. We note that for the Eisenstein series the unfolding provides the natural choice of the basis and hence the normalization of functions $F_\tau(n,z)$. In particular, we can choose the same normalization for cusp forms as well.

The same is true for special functions $G_\tau(\chi,z)$ appearing in the hyperbolic expansion (1.6). Functions $G_\tau(\chi,z)$ could be defined via the generalized matrix coefficient $G_\tau(\bar{\chi},g)=\langle\pi_\tau(g)e_0,d_\chi\rangle$, where d_χ is an explicit χ-equivariant functional on the representation V_τ (e.g., see Section 3.3). It is easy to write down explicitly such a functional in one of the models of the representation π_τ. For example, in a line model such a functional is given essentially by the character χ itself, twisted by τ in order to compensate for the action of A in the line model of π_τ. Hence in such a realization the functional d_χ is given by the (shifted) Mellin transform.

2.4. *Automorphic functionals*

We now switch to a more classical language of automorphic representations of $G_{\mathbb{R}} = \mathrm{PGL}_2(\mathbb{R})$. Let $\Gamma = \mathrm{PGL}_2(\mathbb{Z})$ and $X_{\mathbb{R}} = \Gamma \setminus G_{\mathbb{R}}$. We will view automorphic representations through the Frobenius reciprocity (see [BR]).

2.4.1. *Cusp forms*

Let $e \in X_{\mathbb{R}}$ be the class of the identity element. Evaluation at this point defines a Γ-invariant functional on the space of smooth functions $C^\infty(X_{\mathbb{R}})$. Let (π, V_π, ν_π) be an automorphic cuspidal representation, where V_π is the space of smooth vectors of π and $\nu_\pi : V_\pi \to L^2(X)$ is an isometry. It is well-known that $\nu_\pi : V_\pi \to C^\infty(X_{\mathbb{R}})$. Hence we obtain the Γ-invariant functional $\ell_\nu \in \mathrm{Hom}_\Gamma(V_\pi, \mathbb{C})$ given by $\ell_\nu(v) = \nu_\pi(v)(e)$ for any $v \in V_\pi$. The Frobenius reciprocity of Gelfand and Fomin [G6] (also see [BR] for the quantitative version) is the isomorphism $\mathrm{Hom}_G(V_\pi, C^\infty(X_{\mathbb{R}})) \simeq \mathrm{Hom}_\Gamma(V_\pi, \mathbb{C})$. Given $\ell \in \mathrm{Hom}_\Gamma(V_\pi, \mathbb{C})$ we obtain $\nu_\ell : V_\pi \to C^\infty(X_{\mathbb{R}})$ by $\nu_\ell(v)(g) = \ell(\pi(g)v)$. It is also well-known that a cuspidal $\nu_\pi : V_\pi \to C^\infty(X_{\mathbb{R}})$ extends to the map of Hilbert spaces $\nu_\pi : L_\pi \to L^2(X_{\mathbb{R}})$, where L_π is the completion of V_π with respect to the invariant Hermitian norm.

2.4.2. *Eisenstein series*

Let $E_s(g)$ be the normalized non-holomorphic Eisenstein series for $\mathrm{PGL}_2(\mathbb{Z})$ as in Section 2.1. The theory of (smooth) Eisenstein series implies that the function $E_s(g)$ generates an irreducible (for $s \neq 1$) smooth representation $Eis_s \subset C^\infty(X_{\mathbb{R}})$ which is isomorphic to the (generalized) principal series representation π_{1-2s}. Hence the evaluation at $e \in X_{\mathbb{R}}$ defines a Γ-invariant functional $\ell_{2s-1} \in \mathrm{Hom}_\Gamma(\pi_{1-2s}, \mathbb{C})$. The automorphic function (i.e., the automorphic realization) ϕ_v corresponding to a vector $v \in V_{\pi_{1-2s}}$ is given by $\phi_v(x) = \ell_{2s-1}(\pi_{1-2s}(g)v)$.

It is natural to view the functional ℓ_{2s-1} as a (generalized) vector in the dual representation π_{2s-1} (i.e., in the usual notations $\ell_{2s-1} \in V^{-\infty}_{\pi_{2s-1}}$). We have the canonical pairing $\langle \cdot, \cdot \rangle : \pi_{1-2s} \otimes \pi_{2s-1} \to \mathbb{C}$. We assume that this pairing coincides with the pairing on automorphic functions given by the integral over $X_{\mathbb{R}}$. Hence we have $\langle \ell_{2s-1}, v \rangle = E_s(v, g)|_{g=e}$ for a vector $v \in V_{\pi_{1-2s}}$.

2.4.3. *Hecke operators*

We consider Hecke operators acting on automorphic representations of $G_{\mathbb{R}}$. The theory of Hecke operators provides for each integer prime p, a collection of elements $\gamma_i \in \mathrm{PGL}_2(\mathbb{Q})$, $0 \leqslant i \leqslant p$, such that the Hecke operator acting on the space $C^\infty(X_{\mathbb{R}})$ is given by

$$T_p(f)(x) = \frac{1}{\sqrt{p}} \sum_i f(\gamma_i x). \tag{2.13}$$

The Eisenstein series $E_s(g)$ is an eigenvector of an operator T_p with the eigenvalue $\lambda_p(s) = p^{\frac{1}{2}-s} + p^{s-\frac{1}{2}}$. For a cuspidal representation (π, ν), we denote the corresponding eigenvalues by $\lambda_p(\pi)$ suppressing the dependence on ν (in fact, if π stands for a representation of the adele group, then the multiplicity one for automorphic representations of PGL_2 implies that π determines the image of ν uniquely).

The operator T_p is a scalar operator on the space Eis_s (or in fact on any automorphic representation of $G_{\mathbb{R}}$ coming from an adele automorphic representation). It turns out that, as a result, the functional ℓ_{2s-1} is an eigenfunctional of some operators with the same eigenvalue $\lambda_p(s)$ (or rather with $\lambda_p(1-s) = \lambda_p(s)$) with respect to the usual action on the right by $G_{\mathbb{R}}$ on the automorphic representation Eis_{1-s} (the dual of Eis_s). We emphasize that there is no group algebra "action" of T_p on π. It is acting as a *scalar* operator on the automorphic realization of π! The formula (2.13) does not come from the group algebra action of $\mathrm{PGL}_2(\mathbb{R})$ on $X_{\mathbb{R}}$, but from the action of the adele group. However, on the special vector ℓ, this scalar action *coincides* with the action of an operator coming from the *group algebra action* of $\mathrm{PGL}_2(\mathbb{R})$ on π.

2.4.4. Hecke and Frobenius

Let $\nu : V_\pi \to C^\infty(X)$ be an automorphic representation. Hence a vector $v \in V_\pi$ (in an abstract representation π) has the corresponding automorphic realization $\phi_v(x) = \nu(\pi(g)v) \in C^\infty(X)$. Let $\ell_\nu \in \mathrm{Hom}(\pi, \mathbb{C})$ be the corresponding automorphic functional given by $\ell_\nu(v) = \nu(v)(x)|_{x=e}$ for any smooth vector $v \in V_\pi$ in π. We write ℓ_π suppressing ν and view it as a (generalized) vector in the dual representation π^*. We denote by $\langle\cdot,\cdot\rangle : \pi \otimes \pi^* \to \mathbb{C}$ the natural pairing. We assume it coincides with the pairing on X for automorphic realizations of π and π^* (at least for cuspidal ν).

It turns out that the functional ℓ_π is an eigenfunction of some similarly looking operators with the same eigenvalue $\lambda_p(\pi)$ with respect to the *usual* action of $G_{\mathbb{R}}$ on the right on functions on $X_{\mathbb{R}}$.

Consider elements $\gamma_i \in G_{\mathbb{R}}$ from the expression (2.13) for Hecke operators. Let $\mathcal{T}_p = \frac{1}{\sqrt{p}} \sum_i \gamma_i^{-1}$ be an element of the group algebra of $G_{\mathbb{R}}$. We want to show that $\pi^*(\mathcal{T}_p)\ell_\pi = \lambda_p(\pi) \cdot \ell_\pi$. We have

$$\begin{aligned}\big\langle \pi^*(\mathcal{T}_p)\ell_\pi, v\big\rangle &= \frac{1}{\sqrt{p}} \sum_i \big\langle \pi^*\big(\gamma_i^{-1}\big)\ell_\pi, v\big\rangle = \frac{1}{\sqrt{p}} \sum_i \big\langle \pi(\gamma_i)^*\ell_\pi, v\big\rangle \\ &= \frac{1}{\sqrt{p}} \sum_i \big\langle \ell_\pi, \pi(\gamma_i)v\big\rangle = \left[\frac{1}{\sqrt{p}} \sum_i \nu(v)(x\gamma_i)\right]\bigg|_{x=e} \\ &= \big[T_p\big(\nu(v)\big)(x)\big]\big|_{x=e} = \lambda_p(\pi)\cdot\nu(v)(e) = \lambda_p(\pi)\cdot\langle \ell_\pi, v\rangle\end{aligned}$$

for any $v \in V_\pi$. Hence we have

$$\pi^*(\mathcal{T}_p)\ell_\pi = \frac{1}{\sqrt{p}} \sum_i \pi^*(\gamma_i)\ell_\pi = \lambda_p(\pi)\cdot\ell_\pi. \tag{2.14}$$

We stress again that this is *not* the action of the classical Hecke operator on the automorphic representation (π, ν) (since T_p acts by the scalar $\lambda_p(\pi)$ on the whole space V_π).

We also have (essentially from the definition)

$$\big\langle \pi^*(\mathcal{T}_p)\ell_\pi, v\big\rangle = \big\langle \ell_\pi, \pi\big(\mathcal{T}_p'\big)v\big\rangle, \tag{2.15}$$

where $\pi(\mathcal{T}_p') = \frac{1}{\sqrt{p}} \sum_i \pi(\gamma_i)$ is the action of an element in the group algebra of $G_{\mathbb{R}}$.

2.5. Approximate eigenvectors

It will be crucial for us that all elements γ_i appearing in the description (2.13) of Hecke operators could be chosen in the same Borel subgroup. The (convenient for us) classical choice for these elements is $\gamma_i = \left(\begin{smallmatrix} p & 0 \\ i & 1\end{smallmatrix}\right)$ for $0 \leqslant i \leqslant p-1$ and $\gamma_p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p\end{smallmatrix}\right)$.

The main observation is that all elements γ_i preserve the point $\theta = 0$ (under the natural action on S^1), the singularity of the vector u_w. As a result, the vector u_w is essentially an eigenvector of the operator $\mathcal{T}_p'$. We have the following elementary

Lemma. *Let π_τ be a principal series representation of* $\mathrm{PGL}_2(\mathbb{R})$*, p be an integer prime, elements γ_i as above, and $\mathcal{T}_p' = p^{-\frac{1}{2}} \sum \gamma_i$ be the corresponding element in the group algebra. For any $\sigma \in \mathbb{C}$ and a smooth function $g \in C^\infty(S^1)$ with a small enough support around $\theta = 0$, the following relation holds*:

$$\pi_\tau\big(\mathcal{T}'_p\big)\big(|\theta|^\sigma \cdot g(\theta)\big) = \beta_p(\tau,\sigma)\cdot|\theta|^\sigma g_{p,\tau,\sigma}(\theta), \tag{2.16}$$

where functions $g_{p,\tau,\sigma}$ are smooth functions (in θ) holomorphically depending on τ and σ, and $\beta_p(\tau,\sigma) = p^{\frac{\tau}{2}-\sigma} + p^{\sigma-\frac{\tau}{2}}$. Moreover, we have $g_{p,\tau,\sigma}(0) = g(0)$ for all τ and σ.

Here we view all functions of the variable θ (possibly depending on complex parameters τ and σ) as (a family of) vectors in the representation π_τ (realized in the same model space $C^\infty_{ev}(S^1)$, but with the action of $\mathrm{PGL}_2(\mathbb{R})$ depending on τ).

Proof of the lemma. It is easier to write formulas in the line (or unipotent) model of π_τ. We recall that the principal series representation π_τ of $\mathrm{PGL}_2(\mathbb{R})$ with the trivial character (i.e., representation of $\mathrm{PGL}_2(\mathbb{R})$) has a realization in the space of functions on the real line, and the action is given by restricting the action $\pi_\tau(g)f(x) = f(g^{-1}x)|\det g|^{\frac{\tau-1}{2}}$, $x \in \mathbb{R}^2$, on the plane to the line $\{x = (t,1)\}$. Specializing to the (lower) Borel subgroup, we have

$$\pi_\tau\left(\begin{pmatrix} a & 0 \\ b & c\end{pmatrix}\right) f(t) = f\left(\frac{ct}{-bt+a}\right)|-bt+a|^{\tau-1}|ac|^{\frac{1-\tau}{2}}.$$

Consider a vector $v(\theta) = |\theta|^\sigma \cdot g(\theta)$, $\theta \in S^1$, in the circle model of the representation π_τ. Assume that g is a smooth function and has small support around $\theta = 0$ (and hence $|\theta|^\sigma$ makes sense). Clearly, in the line model such a vector is given by $v(t) = |t|^\sigma \tilde{g}_{\tau,\sigma}(t)$ for some smooth function $\tilde{g}_{\tau,\sigma}$ supported near $t = 0$, and depending holomorphically on τ and σ.

Hence for γ_i as above and $f \in C^\infty(\mathbb{R})$ supported in a small enough neighborhood of $0 \in \mathbb{R}$, we have $\pi_\tau(\gamma_i)(|t|^\sigma f(t)) = p^{-\sigma+\frac{\tau-1}{2}}|t|^\sigma f_{\tau,\sigma,i}(t)$ for $0 \leqslant i \leqslant p-1$, and $\pi_\tau(\gamma_p)(|t|^\sigma f(t)) = p^{\sigma-\frac{\tau-1}{2}}|t|^\sigma f_{\tau,\sigma,p}(t)$, where functions $f_{\tau,\sigma,i}$ are smooth compactly supported functions on $\mathbb{R}$ (depending on τ, σ and γ_i). Hence we have $\pi_\tau(\mathcal{T}'_p)(|t|^\sigma f(t)) = (p^{-\sigma+\frac{\tau}{2}} + p^{\sigma-\frac{\tau}{2}})|t|^\sigma g_{p,\tau,\sigma}(t)$ for some smooth function $g_{p,\tau,\sigma}$. Taking the limit $t \to 0$ on both sides, we obtain the last claim in the lemma. $\square$

3. Meromorphic continuation

3.1. We have the following main result

Theorem. *Let $E(s,z)$ be the classical (normalized) Eisenstein series for $\Gamma = \mathrm{PGL}_2(\mathbb{Z})$ and ℓ_{2s-1} the corresponding automorphic functional on the irreducible representation π_{1-2s} of the principal series of $\mathrm{PGL}_2(\mathbb{R})$. Let $u_{w,z} \in V_{\pi_{1-2s}}$ be a vector such that in the line model of π_{1-2s} it is given by $u_{w,z}(t) = |t|^{w-1}F_z(t)$, where $w \in \mathbb{C}$, $Re(w) \gg 1$, and $F_z \in C^\infty(\mathbb{R})$, is a holomorphic family (in z) of smooth functions of compact support. Then the function defined for $Re(w) \gg 1$, by $\ell_{2s-1}(u_{w,z})$, is a meromorphic function in s, w and z.*

The same claim holds for the function $\ell_\pi(u_{w,z})$ associated to a Hecke–Maass cuspidal representation π.

In other words, the value at the identity for the Eisenstein series $E(s,u_{w,z},g)$ is a meromorphic function in s, w and z. The same holds for the cuspidal function $\phi_{u_{w,z}}(g)$ evaluated at $g = e$.

Proof of the theorem. We first treat the case of cusp forms and then discuss a more delicate case of the Eisenstein series. The main difference between these two cases concerns the issue of boundness of the corresponding automorphic functional in an appropriate norm. For cusp forms, there is a clear answer in terms of Sobolev norms provided by [BR]. For the Eisenstein series, we will use the Fourier expansion instead.

Let $\{F_z\}_{z\in Z}$ be an analytic family of compactly supported smooth functions on $\mathbb{R}$, and $u_{w,z}(t) = |t|^{w-1}F_z(t)$ the corresponding family of functions which we view as an analytic family of vectors in the line model of an appropriate representation of the principal series. Consider an automorphic cuspidal representation $\pi \simeq \pi_\tau$ of the principal series, and the corresponding automorphic functional ℓ_π. The main theorem of [BR] claims that the functional ℓ_π belongs to some Sobolev space completion of the dual representation $\pi^* \simeq \pi_{-\tau}$. This implies that the value of the corresponding automorphic function at the identity, which is given by the pairing

$$\phi_{u_{w,z}}(e) = \big\langle \ell_\pi, |t|^{w-1} F_z(t)\big\rangle,$$

is well-defined for $Re(w) \geqslant T$ for some $T > 0$ which is large enough. Moreover, the function $\phi_{u_{w,z}}(e)$ is analytic in parameters $w \in \mathbb{C}$ and $z \in Z$ wherever it is well-defined. In fact, we can choose $T = 1$ since the above quoted theorem from [BR] states that ℓ_π is bounded in the L^2-Sobolev norm of index σ for any $\sigma > 1/2$.

Consider the operator $\mathcal{T}'_p$ from Lemma 2.5, and the function

$$g_w(t) = \beta_p(\tau, w-1) \cdot u_{w,z} - \pi_\tau\big(\mathcal{T}'_p\big)(u_{w,z}).$$

This is an analytic family of vectors in the space V_{π_τ}. Lemma 2.5 implies (via the computation of the germ at $t = 0$) that $g_w(t) = |t|^w \tilde{g}_{p,\tau,w,z}(t)$, where $\tilde{g}_{p,\tau,w,z}$ is a smooth compactly supported function analytically depending on all parameters. Hence the function g_w belongs to the Sobolev space on which the functional ℓ_π is well-defined for w in a bigger region $Re(w) \geqslant T - 1$. This implies the meromorphic continuation of $\langle \ell_\pi, u_{w,z}\rangle$ to a bigger strip. Namely, it follows from (2.15) and (2.14) that

$$\begin{aligned}
\langle \ell_\pi, g_w\rangle &= \big\langle \ell_\pi, \beta_p(\tau, w-1) \cdot u_{w,z} - \pi_\tau\big(\mathcal{T}'_p\big)(u_{w,z})\big\rangle \\
&= \beta_p(\tau, w-1)\langle \ell_\pi, u_{w,z}\rangle - \big\langle \pi_{-\tau}(\mathcal{T}_p)\ell_\pi, u_{w,z}\big\rangle \\
&= \big[\beta_p(\tau, w-1) - \lambda_p(\pi)\big] \cdot \langle \ell_\pi, u_{w,z}\rangle.
\end{aligned}$$

The left hand side is defined for $Re(w) > T - 1$. Hence we obtain the meromorphic continuation of $\langle \ell_\pi, u_{w,z}\rangle$ to the half-plane $Re(w) > T - 1$ which is to the left of the half-plane $Re(w) \geqslant T$ where $\langle \ell_\pi, u_{w,z}\rangle$ was originally defined. This continuation is defined off the zero set of the function

$$b_p(\tau, w-1) = \beta_p(\tau, w-1) - \lambda_p(\tau) = p^{1-w+\frac{\tau}{2}} + p^{-1+w-\frac{\tau}{2}} - \lambda_p(\tau).$$

However, if $w_0 \in \mathbb{C}$ is a zero of $b_p(\tau, w-1)$ for a given p, we can change the prime p. For cuspidal representations, it is well-known that not *all* eigenvalues of Hecke operators are of the form $\lambda_p(\pi) = p^\lambda + p^{-\lambda}$ for the *same* $\lambda \in \mathbb{C}$ independent of p, and hence the above argument shows that $\tilde{D}_E(\phi, w)$ is holomorphic.

For the Eisenstein series, the treatment is in principle identical. The only issue we have to resolve is the existence of an appropriate norm on the representation π_{2s-1} with respect to which the functional ℓ_{2s-1} is bounded. Results form [BR] are not directly applicable in this case since it was required there that the representation appear discretely in $L^2(\Gamma \setminus G)$. One can however deduce from the theory of smooth Eisenstein series (e.g., [BK] and [L]) that the functional ℓ_{2s-1} is bounded in a smooth enough Sobolev norm. A more elementary treatment is also available from the Fourier expansion of the Eisenstein series (e.g., from the fact that Fourier coefficients are at most polynomial for fixed s). Hence the value

$$E\big(s, |t|^{w-1} F_z(t), e\big) = \big\langle \ell_{2s-1}, |t|^{w-1} F_z(t)\big\rangle$$

is well-defined for $Re(w) \geqslant T(s)$ with $T(s)$ depending on s. The rest of the argument goes as in the cuspidal case, and gives the meromorphic continuation of $\tilde{D}_E(s, w)$. This continuation is defined off the zero set of the function

$$\begin{aligned} b_p(1-2s, w-1) &= \beta_p(1-2s, w-1) - \lambda_p(s) \\ &= p^{w+s-1/2} + p^{-w-s+1/2} - \left(p^{s-1/2} + p^{1/2-s}\right). \end{aligned}$$

Values of w which are zeros of all functions $b_p(1-2s, w-1)$ for all primes p are $w=1$ and $w=2-2s$. However, once w_0 is a potential pole, all values $w_0 - j$, $j=0,1,2,\dots$ are potential poles due to the iterative process of the continuation strip by strip. Hence the potential polar divisor of $\tilde{D}_E(s,w)$ is contained in the set

$$w = 1 - j \quad \text{and} \quad w = 2 - 2s - j, \quad \text{for } j = 0, 1, 2, \dots. \tag{3.1}$$

Using the Fourier expansion for the Eisenstein series, one can see that there are in fact poles at $w = 1, 2-2s$. $\square$

Remark. One can see that the main property of the test vector u_w we use is that it is essentially an eigenvector in V_{π_τ} for a Borel subgroup. This is achieved by considering a vector which is essentially a small piece of a multiplicative character of a torus (in that Borel subgroup). Hence it would be more natural from the point of view of representation theory to parameterize vectors u_w by that character and not by the “artificial” parameter w appearing in $D_E(s,w)$. This introduces a shift by the infinitesimal parameter of the representation. Namely, u_w corresponds to the character $\mathrm{diag}(a, a^{-1}) \mapsto |a|^{-\alpha}$ of the diagonal subgroup, where $\alpha = -2 + 2s + 2w$. Accordingly, the polar set (3.1) takes a more symmetric form (with respect to the natural change $s \mapsto 1-s$) $\alpha = -2 + 2s - 2j$ and $\alpha = -2s - 2j$, $j = 0, 1, 2, \dots$.

3.2. *General* CM-*points*

Let $\mathfrak{z} \in \mathcal{H}$ be a CM-point corresponding to an imaginary quadratic field E. There exists a non-trivial element $\gamma_{\mathfrak{z}} \in \mathrm{PGL}_2(\mathbb{Q})$ which fixes $\mathfrak{z}$. Consider the connected compact subgroup $K_{\mathfrak{z}} \subset \mathrm{PGL}_2(\mathbb{R})$ fixing $\mathfrak{z}$. We have $\gamma_{\mathfrak{z}} \in K_{\mathfrak{z}}$. Let $h \in \mathrm{PGL}_2^+(\mathbb{R})$ be an element such that $K_{\mathfrak{z}} = h^{-1}\,\mathrm{PSO}(2,\mathbb{R})h$. Consider the set $S_{\mathfrak{z}} = K_{\mathfrak{z}} \cdot (1,0)^t \subset \mathbb{R}^2 \setminus 0$ (i.e., $S_{\mathfrak{z}} = h^{-1}S^1$ for the standard circle S^1). Note that we have a rational point $s_{\mathfrak{z}} = \gamma_{\mathfrak{z}} \cdot (1,0)^t \in S_{\mathfrak{z}}$ on this ellipse. Let $B_{\mathfrak{z}} \subset \mathrm{PGL}_2(\mathbb{Q})$ be the rational Borel subgroup having a rational eigenvector $s_{\mathfrak{z}}$. Now we can repeat our construction of the test vector u_w from Section 2.3.2. We consider the orthonormal basis $\{e_n^{\mathfrak{z}} = \pi(h^{-1})e_n\}_{n \in 2\mathbb{Z}}$ of $K_{\mathfrak{z}}$-types. This allows us to normalize corresponding matrix coefficients by $F_\tau^{\mathfrak{z}}(n,g) = \langle \pi(g)e_0^{\mathfrak{z}}, e_n^{\mathfrak{z}} \rangle_{\pi_\tau}$, and obtain the expansion $\phi(z) = \sum_{n \in 2\mathbb{Z}} a_n^{\mathfrak{z}} F_\tau^{\mathfrak{z}}(n,z)$ analogous to the expansion (1.3) (this is the spherical expansion of ϕ centered at $\mathfrak{z}$). The corresponding test vector is given by $u_w^{\mathfrak{z}} = \pi(h^{-1})u_w$, and as a function on $S_{\mathfrak{z}}$ has the singularity at the point $s_{\mathfrak{z}}$. The proof that the vector $u_w^{\mathfrak{z}}$ is an approximate eigenvector of Hecke operators given in Section 2.5 now proceeds as before once we notice that one can choose representatives γ_i for a Hecke operator in the *rational* Borel subgroup $B_{\mathfrak{z}}$. The rest of the proof of Theorem 3.1 is identical to the case we considered.

3.3. *Real quadratic periods*

Only a slight modification is needed in order to treat real quadratic fields, and, as is well-known, adelically one can treat CM and real quadratic fields simultaneously.

Let $\ell \subset \mathcal{H}$ be a geodesic corresponding to a real quadratic field E (the equivalence class of such geodesics corresponds to an appropriate class group of E). There exists a non-trivial element $\gamma_\ell \in \mathrm{PGL}_2(\mathbb{Q})$ fixing ℓ, which is conjugate to a diagonal matrix $h\gamma_\ell h^{-1} = \mathrm{diag}(u, u^{-1})$, where u is a unit in E. We will assume that γ_ℓ generates the corresponding group of units (i.e., u is a fundamental unit), and choose two linearly independent eigenvectors $v_1, v_2 \in \mathbb{R}^2$ of γ_ℓ (note that $v_i \notin \mathbb{Q}^2$). Consider a character of the diagonal group $\chi_\alpha(\mathrm{diag}(a,b)) = |a/b|^{-\alpha}$. To any such a character, we associate the equivariant functional $d_\alpha : V_{\pi_\tau} \to \mathbb{C}$ given in the plane model by the kernel $d_\alpha(x,y) = |x|^{\alpha+\tau/2-1/2}|y|^{-\alpha+\tau/2-1/2}$ (i.e., $\pi_{-\tau}(\mathrm{diag}(a,b))d_\alpha = \chi_\alpha(\mathrm{diag}(a,b))d_\alpha$). The functional $d_\alpha^\ell = \pi_{-\tau}(h)d_\alpha$

is γ_ℓ-equivariant (i.e., satisfies $\pi_{-\tau}(\gamma_\ell)d_\alpha = \chi_\alpha(u)d_\alpha$). We consider characters $\chi_i = \chi_{\alpha_i}$, $i \in \mathbb{Z}$, which are trivial on the unit group, i.e., $|u|^{\alpha_i} = 1$ (this implies that $\alpha_i = i\alpha_1$). We define then special functions for the hyperbolic expansion (1.6) by $G_\tau(i, g) = \langle \pi_\tau(g)e_0, d_{\chi_i} \rangle$.

Let $B(\mathbb{Q})$ be a rational Borel subgroup, and $\xi \in \mathbb{R}^2 \setminus 0$ be an eigenvector of $B(\mathbb{Q})$. Consider an *affine* line $L \subset \mathbb{R}^2 \setminus 0$ generated by the eigenvector v_1 of γ_ℓ (it is transversal to ξ). We denote by 0_L the point of intersection of L with the line $\mathbb{R}v_2$, and introduce the linear parameter t on L such that $t = 0$ corresponds to 0_L. Let H_{π_τ} be the plane realization of the principal series representation V_{π_τ} (i.e., H_{π_τ} is the space of homogeneous functions of the homogeneous degree $\tau - 1$). We restrict functions in H_{π_τ} to the affine line L, and obtain the standard (twisted) linear fractional action of G on the line model for π_τ. All elements in $B(\mathbb{Q})$ are fixing the point $b_L = L \cap \mathbb{R}\xi$, which we assume corresponds to $t = 1$. Hence we can repeat our construction by choosing Hecke operators with representatives in $B(\mathbb{Q})$, and construct the test vector $u_w(t) = |t-1|^{w-1} f(t-1)$ through the coordinate t as in Section 2.3.2. The computation of the spectral expansion of u_w with respect to d_α^ℓ is straightforward since on L the functional d_α^ℓ coincides with the Mellin transform in t (and hence, $\langle u_w, d_\alpha^\ell \rangle$ is given essentially by the Beta function).

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